



Discrete Mathematics 189 (1998) 283–289

**DISCRETE
MATHEMATICS**

Note

On packing minors into connected matroids¹Manoel Lemos^a, James Oxley^{b,*}^a *Departamento de Matemática, Universidade Federal de Pernambuco, Recife,
Pernambuco 50740-540, Brazil*^b *Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803-4918, USA*

Received 16 July 1996; revised 22 December 1997; accepted 2 February 1998

Abstract

Let N be a matroid with k connected components and M be a minor-minimal connected matroid having N as a minor. This note proves that $|E(M) - E(N)|$ is at most $2k - 2$ unless N or its dual is free, in which case $|E(M) - E(N)| \leq k - 1$. Examples are given to show that these bounds are best possible for all choices for N . A consequence of the main result is that a minimally connected matroid of rank r and maximum circuit size c has at most $2r - c + 2$ elements. This bound sharpens a result of Murty. © 1998 Elsevier Science B.V. All rights reserved

AMS classification: primary 05B35

1. Introduction

For $n \in \{2, 3\}$, let N be a minor of an n -connected matroid M and let M' be a minor of M that is minimal with the properties of being n -connected and having N as a minor. Semple and Whittle (private communication) asked for a bound on $|E(M') - E(N)|$. This paper gives such a bound that is best possible for all choices of N in the case that $n = 2$. The case when $n = 3$ is considerably more difficult and will be treated elsewhere.

If, in the original problem, N itself is n -connected, then clearly $M' = N$. Thus, the problem only becomes non-trivial when N is not n -connected. However, a close relative of the problem, which arises precisely when N is n -connected and has received considerable attention, imposes the additional requirement that $M' \neq N$. When $n = 3$, Truemper [8] solved this variant of the problem by proving that $|E(M') - E(N)| \leq 3$. For

¹ The authors were partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil) and the National Security Agency, respectively.

* Corresponding author. E-mail: oxley@math.lsu.edu.

the case $n=2$, the variant is solved by the following well-known result of Brylawski [2] and Seymour [7] (see also [6, Theorem 4.3.6]). Note that a matroid is connected if and only if it is 2-connected.

Theorem 1.1. *Let N be a connected minor of a connected matroid M and e be an element of $E(M) - E(N)$. Then $M \setminus e$ or M/e is connected and has N as a minor.*

Another natural variant of the original problem is to weaken the condition on M' to the requirement that it is a minimal n -connected minor of M having a minor isomorphic to, rather than equal to, N . This variant will be addressed, again for the case $n=2$, at the end of the next section following the proofs of the main results.

Familiarity will be assumed here with the basics of matroid theory. Our notation and terminology will follow [6]. The proofs will rely on several elementary properties of the operations of series and parallel connection. These may be found in Section 7.1 of [6] or in the original source [1]. We denote the number of connected components of a non-empty matroid M by $\lambda_1(M)$, and let $\lambda_1(U_{0,0}) = 1$.

2. The results

The main result of the paper, which is stated in the abstract, appears as Corollary 2.4 below. It will be proved by first establishing the corresponding theorem in the case when N is a restriction rather than an arbitrary minor of M .

Theorem 2.1. *Let N be a matroid and M be a minor-minimal connected matroid having N as a restriction. Then*

$$|E(M)| - |E(N)| \leq \lambda_1(N) - 1.$$

Proof. If N has a loop, then $N \cong U_{0,1}$ and so $M = N$ and the result holds. Thus, we may assume that N is loopless. Let (M, N) be a counterexample with $|E(M)|$ minimal. Let $A = E(N)$. If $e \in E(M) - \text{cl}_M(A)$, then N occurs as a restriction of both $M \setminus e$ and M/e . Since at least one of the last two matroids is connected, we deduce that the choice of M is contradicted. We conclude that A spans M .

Let $E(M) - A = \{e_1, e_2, \dots, e_k\}$. We show next that, for all i in $\{1, 2, \dots, k\}$,

$$2.2. \lambda_1(M \setminus \{e_1, e_2, \dots, e_{i-1}\}) < \lambda_1(M \setminus \{e_1, e_2, \dots, e_i\}).$$

As A spans M , it is clear that $\lambda_1(M \setminus \{e_1, e_2, \dots, e_{i-1}\}) \leq \lambda_1(M \setminus \{e_1, e_2, \dots, e_i\})$ for all i . Suppose that equality occurs for some i and let M_i be the component of $M \setminus \{e_1, e_2, \dots, e_{i-1}\}$ that contains e_i . Clearly, $M_i \setminus e_i$ is connected but $M \setminus e_i$ is disconnected. Thus, $E(M_i \setminus e_i)$ is contained in some component of $M \setminus e_i$. Since e_i is contained in the closure of this component in M , we deduce that $\lambda_1(M) = \lambda_1(M \setminus e_i)$. But $\lambda_1(M) = 1$ and $\lambda_1(M \setminus e_i) > 1$. This contradiction completes the proof of (2.2).

From (2.2), $\lambda_1(M) \leq \lambda_1(N) - k$. But $\lambda_1(M) = 1$ and $k = |E(M)| - |E(N)|$. The theorem follows immediately. \square

Corollary 2.3. *Let N be a matroid and M be a minor-minimal connected matroid having N as a minor. Then*

$$|E(M)| - |E(N)| \leq 2\lambda_1(N) - 2.$$

Proof. Evidently, $N = M \setminus X/Y$ for some sets X and Y . Let X_1 be a maximal subset of $X \cup Y$ containing X such that $N = M \setminus X_1 / [(X \cup Y) - X_1]$. Let $N_1 = M \setminus X_1$. Then M is a minor-minimal connected matroid having N_1 as a restriction. Thus, by Theorem 2.1,

$$|E(M)| - |E(N_1)| \leq \lambda_1(M \setminus X_1) - 1. \quad (1)$$

Suppose $M \setminus X_1$ has a component whose ground set Z is contained in $(X \cup Y) - X_1$. Then $M \setminus X_1 / Z = M \setminus X_1 \setminus Z$ and so $N = M \setminus (X_1 \cup Z) / [(X \cup Y) - (X_1 \cup Z)]$ and the choice of X_1 is contradicted. Hence, every component of $M \setminus X_1$ contains an element of N . Thus,

$$\lambda_1(M \setminus X_1) \leq \lambda_1(N)$$

and so, from (1),

$$|X| \leq |X_1| = |E(M)| - |E(N_1)| \leq \lambda_1(N) - 1.$$

By applying the above argument using M^* and N^* in place of M and N , respectively, we deduce that

$$|Y| \leq \lambda_1(N) - 1.$$

Hence $|E(M)| - |E(N)| = |X| + |Y| \leq 2\lambda_1(N) - 2$, as required. \square

Corollary 2.4. *Let N be a matroid and M be a minor-minimal connected matroid having N as a minor. Then*

$$|E(M)| - |E(N)| \leq \begin{cases} \lambda_1(N) - 1 & \text{if } r(N) \in \{0, |E(N)|\}, \\ 2\lambda_1(N) - 2 & \text{otherwise.} \end{cases}$$

Moreover, for all choices for N , there is a minor-minimal connected matroid M having N as a minor such that this bound is attained.

Proof. The bound follows immediately by combining Theorem 2.1 and its dual with Corollary 2.3. It remains to show that the bound is attainable. Suppose first that $r(N) = |E(N)|$. Then $N \cong U_{n,n}$ for some non-negative integer n . If $n \leq 1$, then we take $M = N$. If $n \geq 2$, take $M = M(G)$ where G is obtained by joining every vertex of a path P of length $n - 1$ to a new vertex x . Here $N = M \setminus E(P)$ and M is certainly a minor-minimal connected matroid having N as a minor. If $r(N)$ is 0 instead of $|E(N)|$, then the dual of the last example attains the bound.

Now, assume that $r(N) \notin \{0, |E(N)|\}$. We shall prove, by induction on $\lambda_1(N)$, that, for all choices of N , equality is attained by some matroid M . If $\lambda_1(N) = 1$, then we can take $M = N$ to attain the bound. Now, assume that the bound is attainable whenever $\lambda_1(N) < k$ and let $\lambda_1(N) = k \geq 2$. Let $N = N_1 \oplus N_2 \oplus \cdots \oplus N_k$. Then, by switching to the dual or relabelling if necessary, we may assume that either (a) $N \cong N_1 \oplus U_{0,k-1}$ where N_1 is not a loop; or (b) N_1 is not a coloop, N_2 is not a loop, and $N_2 \oplus N_3 \oplus \cdots \oplus N_k$ is not a direct sum of coloops. In case (a), let $E(N) - E(N_1) = \{a_2, a_3, \dots, a_k\}$, let e be a fixed element of $E(N_1)$, and, for all i in $\{2, 3, \dots, k\}$, let N'_i be a rank-2 wheel with rim $\{a_i, b_i\}$ and spokes e and c_i . Let M be the parallel connection of $N_1, N'_2, N'_3, \dots, N'_k$ across the common basepoint e . Then $|E(M)| - |E(N)| = 2\lambda_1(N) - 2$, and it is not difficult to check that M is a minor-minimal connected matroid having N as a minor.

In case (b), by the induction assumption, there is a connected matroid M_2 that is minor minimal having $N_2 \oplus N_3 \oplus \cdots \oplus N_k$ as a minor such that

$$|E(M_2)| - |E(N_2 \oplus N_3 \oplus \cdots \oplus N_k)| = 2(k - 1) - 2.$$

Now, let N'_1 be obtained from N_1 by adding an element x' in series with an arbitrarily chosen element x of N_1 . Moreover, let y be an element of $E(N_2)$. Let M_3 be a triangle with ground set $\{x', y, y'\}$ where $y' \notin E(N'_1) \cup E(M_2)$ and let M be formed as follows: let M_1 be the parallel connection with basepoint x' of N'_1 and M_3 ; then let M be the parallel connection with basepoint y of M_1 and M_2 . Now, M is certainly connected and $M \setminus y' / x' = N_1 \oplus M_2$, so M has N as a minor. It remains to check that M is minor minimal with these properties. Assume the contrary and let $M \setminus S / T$ be a minor-minimal connected matroid having N as a minor where $S \cup T \neq \emptyset$. Since $E(N_1)$ and $E(M_2)$ are contained in different components of both $M \setminus y'$ and M / x' , we deduce that $y' \notin S$ and $x' \notin T$.

We show next that

2.5. $M / y' \setminus x'$ does not have N as a minor.

Assume that $M / y' \setminus x' \setminus X / Y = N$. Since x is a coloop of $(M / y' \setminus x') \setminus E(N_1)$, but x is not a coloop of N_1 , the matroid $M / y' \setminus x'$ has a circuit C such that $x \in C \subseteq E(N_1) \cup Y$. Then, as $M / y' \setminus x' = P((M_1 / y' \setminus x'; y), (M_2; y))$, we deduce that $C = (C_1 \cup C_2) - y$ where C_1 and C_2 are circuits of $M_1 / y' \setminus x'$ and M_2 , respectively, containing y . Thus, $C_2 \subseteq Y$ and so $(M / y' \setminus x') / Y$ has y as a loop; a contradiction to the fact that y is not a loop in N . Hence (2.5) holds.

Since M / y' has $\{x', y\}$ as a circuit, (2.5) implies that M / y' does not have N as a minor. Similarly, (2.5) and the fact that $\{x, y'\}$ is a cocircuit of $M \setminus x'$ imply that $M \setminus x'$ does not have N as a minor. We conclude that $\{x', y'\}$ avoids $X \cup Y$, so $X \cup Y \subseteq E(M_2)$ and

$$M \setminus X / Y = P((M_1; y), (M_2 \setminus X / Y; y)).$$

Thus, $M_2 \setminus X / Y$ is connected. Moreover, since neither $M \setminus x'$ nor M / y' has N as a minor, to obtain N as a minor of $M \setminus X / Y$, we must contract x' and delete y' ; that

is, $N_1 \oplus N_2 \oplus \cdots \oplus N_k$ is a minor of $M \setminus X/Y/x' \setminus y'$. But the last matroid equals $P((M_1/x' \setminus y'; y), (M_2 \setminus X/Y; y))$ which, since $M_1/x' \setminus y'$ has y as a coloop, equals $(M_1/x' \setminus y' \setminus y) \oplus (M_2 \setminus X/Y)$, that is, $N_1 \oplus (M_1 \setminus X/Y)$. Hence $N_2 \oplus N_3 \oplus \cdots \oplus N_k$ is a minor of $M_1 \setminus X/Y$; a contradiction. We conclude that M is a minor-minimal connected matroid having $N_1 \oplus N_2 \oplus \cdots \oplus N_k$ as a minor. Since $|E(M)| - |E(M_2)| - |E(N_1)| = 2$, we conclude that

$$|E(M)| - |E(N)| = 2k - 2$$

and it follows, by induction, that the bound in the corollary is attained for all choices of N . \square

In the last corollary, if we do not insist that M has N itself as a minor but only that M has a minor isomorphic to N , then the same bounds hold. However, analyzing when these bounds can be improved for specific choices of N does not seem straightforward. For instance, it is not difficult to show that if $N \cong U_{m,p} \oplus U_{m,q}$ for some m in $\{1, 2\}$ and some p and q exceeding m , then a minor-minimal connected matroid M having a minor isomorphic to N has at most $|E(N)| + 1$ elements. On the other hand, when $N \cong U_{1,s} \oplus U_{s-1,s}$ for some s exceeding three, the following construction produces a connected matroid M with $|E(N)| + 2$ elements such that M is minor minimal having a minor isomorphic to N : Take a copy of the graph K_4 with vertices labelled 1, 2, 3, and 4; delete the edges 13 and 23; replace the edges 12 and 14 by s_1 and $s - s_1$ parallel edges, respectively, where both s_1 and $s - s_1$ are positive; join 2 and 3 by two internally disjoint paths of lengths s_2 and $s - s_2$ where both s_2 and $s - s_2$ exceed one. The desired matroid is the cycle matroid of the resulting graph.

Next, we describe a large class of matroids N for which the bound in Corollary 2.4 is attained under the weaker hypothesis on M , namely that M is a connected matroid which is minor minimal with the property that it has a minor isomorphic to N .

The matroid M_1 is a *coextension* of the matroid M_2 if $M_2 = M_1/e$. If e is not a loop of M_1 and e is in every dependent flat of M_1 , then M_1 is the *free coextension* of M_2 by e , and e is *cofree* in M_1 . Note that an element is cofree in a matroid if and only if it is free in its dual. Let N_1, N_2, \dots, N_k be 3-connected matroids none of which has a free or a cofree element. Suppose $r(N_1) = \min\{r(N_i) : 1 \leq i \leq k\}$ and let N'_1 be the free extension of N_1 by the element e_1 . For all i in $\{2, 3, \dots, k\}$, let N'_i be the free coextension of N_i by e_i , and let N''_i be the parallel connection of N'_i with a triangle on $\{e_i, f_i, e_1\}$ across the basepoint e_i . Finally, let M be obtained by taking the parallel connection of all of $N'_1, N''_2, N''_3, \dots, N''_k$ across the common basepoint e_1 and then deleting e_1 . Equivalently, M is the 2-sum of $N'_1, N''_2, N''_3, \dots, N''_k$ across the common basepoint e_1 . Then $|E(M)| - |E(N_1 \oplus N_2 \oplus \cdots \oplus N_k)| = 2(k - 1)$. Moreover, it can be shown that M is a minor-minimal connected matroid having a minor isomorphic to $N_1 \oplus N_2 \oplus \cdots \oplus N_k$, but we omit the details.

We conclude the paper by proving another consequence of Theorem 2.1, a bound on the maximum number of elements in a minimally connected matroid.

Corollary 2.6. *Let M be a minimally connected matroid whose largest circuit has $c(M)$ elements. Then*

$$|E(M)| \leq 2r(M) - c(M) + 2.$$

Proof. Let C be a circuit of M with $c(M)$ elements. Choose an element e of C and extend $C - e$ to a basis of M by adjoining a set X of elements. Let $N = M|(C \cup X)$. Then M is a connected matroid that is minor minimal having N as a restriction. Thus, by Theorem 2.1, $|E(M)| \leq |E(N)| + \lambda_1(N) - 1$. But $|E(N)| = r(M) + 1$ and $\lambda_1(N) = 1 + (r(M) - (c(M) - 1))$. The corollary follows immediately. \square

It is well known and easy to see that a minimally connected matroid cannot have a triangle unless that triangle is the ground set of the matroid. Hence, if M is a minimally connected matroid of rank at least three, then the last corollary implies that $|E(M)| \leq 2r(M) - 2$. This bound was proved by Murty [4] as a generalization of Dirac's result [3] that a minimally 2-connected graph G with at least four vertices has at most $2|V(G)| - 4$ edges. When the last corollary is applied to graphs, we immediately obtain the following sharpening of Dirac's result.

Corollary 2.7. *Let G be a minimally 2-connected graph with circumference c . Then*

$$|E(G)| \leq 2|V(G)| - c.$$

Finally, we show that the last two corollaries are best possible using a single example. Let m and n be arbitrary integers where $m \geq 2$ and $n \geq 1$. Let G be the graph that consists of two vertices joined by $n + 1$ internally disjoint paths, one of length m and the remaining n of length two. Clearly, G has $n + m + 1$ vertices, $2n + m$ edges, and circumference $m + 2$. Thus it attains the bound in the last corollary, and its cycle matroid attains the bound in Corollary 2.6.

Acknowledgements

The authors thank an anonymous referee for suggesting the current proof of Theorem 2.1 which is significantly shorter than our original proof.

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